ON THE SOLUTION OF PRANDTL EQUATIONS BY THE METHOD OF FINITE DIFFERENCES

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We present two finite difference methods for numerical solution of a system of boundary layer equations for a nonsteady flow of a viscous incompressible fluid. One of these methods is explicit, while the other is implicit and we find, that, in order to obtain an approximate solution by the implicit method we must also solve a linear algebraic system. It is shown that while the explicit difference method is convergent when some constraints are imposed on the relations connecting the steps in spatial and temporal coordinates, the implicit method is convergent without the above constraints. Convergence of these difference methods is proved under the assumption of existence of a smooth solution of a system of boundary layer equations (see [1]).

1. Statement of the problem. Existence of a smooth solution of the system of boundary layer equations for a plane nonsteady flow of a viscous incompressible fluid

$$u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \qquad u_x + v_y = 0 \tag{1.1}$$

in the region $D\{0 \leqslant t < t_0,\ 0 \leqslant x < x_0,\ 0 \leqslant y < \infty\}$ with the conditions

$$u|_{t=0} = u_0(x, y), \qquad u|_{y=0} = 0, \qquad v|_{y=0} = v_0(t, x), \qquad u|_{x=0} = u_1(t, y) \tag{1.2}$$

$$\lim_{y\to\infty} u(t, x, y) = U(t, x) \tag{1.3}$$

was proved in [1] under the assumption that either t_0 or x_0 are not greater than some constants depending on the parameters of the problem (1.1) to (1.3) and under the usual assumption of the smoothness and compatibility of the functions entering the conditions (1.2) and (1.3). By the Bernoulli's law we have

$$-p_x = U_t + UU_x$$

Solution of the problem (1.1) to (1.3) is, in [1], reduced by the change of variables $\tau = t$, $\xi = x$ and $\eta = u(t, x, y)$ and introduction of a new unknown function $w = u_y$, to solution of Equation

$$\nu w^2 w_{\eta\eta} - w_\tau - \eta w_\xi + p_x w_\eta = 0 \tag{1.4}$$

in the region $\Omega\{0 \leqslant \tau < t_0,\ 0 \leqslant \xi < x_0,\ 0 \leqslant \eta < U(\tau, \xi)\}$ with the following conditions

$$w|_{\tau=0} = u_{0y} \equiv w_0(\xi, \eta), \qquad w|_{\xi=0} = u_{1y} \equiv w_1(\tau, \eta), \qquad w|_{\eta=U(\tau, \xi)} = 0 \tag{1.5}$$

$$\nu\ ww_\eta - p_x - v_0 w = 0 \qquad \text{for } \eta = 0 \tag{1.6}$$

Function u entering the solution of (1.1) to (1.3) can be obtained from

$$y = \int_0^{u(t,x,y)} \frac{ds}{w(t,x,s)} \tag{1.7}$$

Below we shall give two finite difference methods for numerical solution of the problem (1.4) to (1.6). Obviously, approximate values can be easily obtained for u using approximate values of w together with the relation of the type of (1.7) the right-hand side of which defines the inverse function of u. These methods are given in [2].. Convergence of approximations obtained by the method of straights for the solutions of steady and nonsteady Prandtl system can be proved in the analogous manner.

We shall assume t_0 and x_0 finite, without loss of generality.

2. Explicit finite difference method. Let a net whose nodes are given by the intersections of planes $\tau = mh$, $\xi = l\sigma$ and $\eta = k\sigma$ $(m, l, k = 0, 1, 2,...)$ where $h > 0$ and $\sigma > 0$ are some constants, be given in the τ, ξ, η-space. We shall call the nodal points

$$(m_1 h,\ l_1\sigma,\ k_1\sigma), \quad (m_1 h,\ (l_1 - 1)\sigma,\ k_1\sigma)$$

$$(m_1 h,\ l_1\sigma,\ (k_1 - 1)\sigma), \quad (m_1 h,\ l_1\sigma,\ (k_1 + 1)\sigma)$$

the neighboring points of the node $((m_1 + 1)h,\ l_1\sigma,\ k_1\sigma)$. Also, we shall denote the set of points belonging to Ω together with its boundaries by Ω' and we shall call the node belonging to Ω' internal if all its neighboring points belong to Ω'. The remaining nodes belonging to Ω' shall be called boundary nodes.

The value of the function f at the node $(mh,\ l\sigma,\ k\sigma)$ will be denoted by f_{mlk} and we shall construct a finite difference equation approximating (1.4) for the function w

$$(\nu w^2_{mlk} + M\sigma)\frac{w_{ml,\,k+1} - 2w_{mlk} + w_{ml,\,k-1}}{\sigma^2} - \frac{w_{m+1,\,lk} - w_{mlk}}{h} -$$

$$- k\sigma\frac{w_{mlk} - w_{m,\,l-1,\,k}}{\sigma} + p_{xmlk}\frac{w_{mlk} - w_{ml,\,k-1}}{\sigma} = 0 \tag{2.1}$$

at each internal point of Ω' with coordinates $((m + 1)h,\ l\sigma,\ k\sigma)$. Here M is a positive constant and $M > \max|p_x|$.

We assume for all the boundary nodes of Ω' lying on the planes $\tau = 0$, $\xi = 0$ and $\eta = 0$ that at these nodes

$$w_{0lk} = w_0(l\sigma,\ k\sigma), \qquad w_{m0k} = w_1(mh,\ k\sigma) \tag{2.2}$$

$$\nu w_{ml0}\frac{w_{m+1,\,l1} - w_{m+1,\,l0}}{\sigma} - p_{x\,ml0} - v_{0ml0}\,w_{ml0} = 0 \tag{2.3}$$

while at the remaining boundary nodes of Ω' denoting them by $\Gamma_{h\sigma}$ we assume that

$$w_{mlk} = 0 \tag{2.4}$$

Equations (2.2) to (2.4) approximate the boundary conditions (1.5) and (1.6). It is clear that when $w_{ml0} \neq 0$ at all boundary nodes of Ω' lying on the plane $\eta = 0$, then the values $w_{m+1,lk}$ (with fixed $m \geq 0$) are uniquely determined by the system of equations (2.1) to (2.4) in terms of w at $\tau = mh$.

To show that the values of w_{mlk}, defined by the difference equations (2.1) to (2.4) at the boundary nodes of Ω' converge to the solution of the problem (1.4) to (1.6) as $h \to 0$ and $\sigma \to 0$, we shall have to prove some auxilliary assumptions. Everywhere in the following M_1 will denote positive constants defined by the parameters of the problem (1.4) to (1.6) and independent of h and σ, and we shall also use the following notation

$$L_{m+1}(z) \equiv [\nu (w_{mlk})^2 + M\sigma] \frac{z_{ml,k+1} - 2z_{mlk} + z_{ml,k-1}}{\sigma^2} -$$

$$- \frac{z_{m+1,lk} - z_{mlk}}{h} - k\sigma \frac{z_{mlk} - z_{m,l-1,k}}{\sigma} + p_{xmlk} \frac{z_{mlk} - z_{ml,k-1}}{\sigma}$$

$$\lambda_{m+1}(z) \equiv \nu w_{ml0} \frac{z_{m+1,l1} - z_{m+1,l0}}{\sigma} - p_{xml0} - v_{0ml0} w_{ml0}$$

Lemma 1. Let function w be given at the nodes of Ω' and let it satisfy the difference equations (2.1) to (2.4) and let the functions F and F_1 be such, that

$$F \leqslant w \leqslant F_1 \tag{2.5}$$

at nodes of Ω', for which $\tau = mh$, when $l = 0$ and on $\Gamma_{h\sigma}$, when $\tau = (m+1)h$. Here $m \geq 0$ is a fixed integer.

We shall assume that $L_{m+1}(F) \geqslant 0$ and $L_{m+1}(F_1) \leqslant 0$ for all k and l corresponding to the internal nodes of Ω' lying on the plane $\tau = (m+1)h$, that $\lambda_{m+1}(F) > 0$ and $\lambda_{m+1}(F_1) < 0$, for l corresponding to the boundary nodes of Ω' of the type $((m+1)h, l\sigma, 0)$ and also that $w_{m0} \neq 0$.

Then, the inequalities (2.5) will hold at all nodes of Ω' for which $\tau = (m+1)h$, provided that $h/\sigma^2 < 1/2\nu a^2$ where $a^2 = \max F_1^2$ when $\tau = mh$.

Proof. We shall first prove that $z = w - F \geq 0$ when $\tau = (m+1)h$. By the previous assumption $z \geq 0$ when $\tau = mh$, on $\Gamma_{h\sigma}$ and for $\xi = 0$ when $\tau = (m+1)h$. By the condition (2.3) and the inequality $\lambda_{m+1}(F) > 0$

$$\nu w_{ml0} \frac{z_{m+1,l1} - z_{m+1,l0}}{\sigma} < 0 \tag{2.6}$$

from which it follows that

$$z_{m+1,l0} > z_{m+1,l1} \tag{2.7}$$

For the internal nodes of Ω' for which $\tau = (m+1)h$ we have $L_{m+1}(w) - L_{m+1}(F) \leqslant 0$. This means that

$$[\nu (w_{mlk})^2 + M\sigma] \frac{z_{ml,k+1} - 2z_{mlk} + z_{ml,k-1}}{\sigma^2} - \frac{z_{m+1,lk} - z_{mlk}}{h} -$$

$$- k\sigma \frac{z_{mlk} - z_{m,l-1,k}}{\sigma} + p_{xmlk} \frac{z_{mlk} - z_{ml,k-1}}{\sigma} \leqslant 0$$

from which we obtain

$$z_{m+1,lk} \geqslant \left(1 - \frac{2\nu (w_{mlk})^2 + 2M\sigma - p_{xmlk}\sigma + k\sigma^2}{\sigma^2} h\right) z_{mlk} + \frac{\nu (w_{mlk})^2 + M\sigma}{\sigma^2} h z_{ml,k+1} +$$

$$+ \left(\frac{\nu (w_{mlk})^2 + M\sigma}{\sigma^2} h - \frac{p_{xmlk}}{\sigma} h\right) z_{ml,k-1} + khz_{m,l-1,k} \tag{2.8}$$

Since by the previous assumption $z_{mlk} \geqslant 0$, it follows from (2.8) that $z_{m+1lk} \geqslant 0$ at all internal nodes of Ω' provided that all the coefficients of z in the right-hand side of (2.8) are nonnegative. Coefficient of $z_{ml,k-1}$ is obviously nonnegative since $M > \max |p_x|$. Coefficient of z_{mlk} in (2.8) will be nonnegative if

$$\frac{h}{\sigma^2} \leqslant \frac{1}{2\nu (w_{mlk})^2 + 2M\sigma - p_{xmlk}\sigma + k\sigma^2} \tag{2.9}$$

The latter obviously holds for sufficiently small σ since by the previous assumption

$h/\sigma^2 < \frac{1}{2}\nu a^2$. Consequently $z_{m+1lk} \geqslant 0$ at all internal nodes of Ω' when $1 = 0$ and on $\Gamma_{h\sigma}$. Hence, from (2, 7) it follows that $z_{m+1l0} \geqslant 0$ and, consequently, $z = w - F \geq 0$ at all nodes of Ω' for which $\tau = (m+1)h$.

The inequality $w - F_1 \leq 0$ is proved analogously.

Lemma 2. At the nodes of Ω' for which $\tau \leq \tau_0$ and at sufficiently small h and σ, the estimates

$$V - v_1 \leqslant w \leqslant V_1 \tag{2.10}$$

hold for the solution w of difference equations (2, 1) to (2, 4). Here V and V_1 are functions constructed in the proof of Lemma 2 of [1], τ_0 is a constant also given in [1], function $v_1 = M_1(\tau + 1)(h + \sigma)$, $h / \sigma^2 < 1/2\, \nu b_1^2$ and $b_1^2 = \max V_1^2$.

At the nodes of Ω' for which $\xi \leq \xi_0$ and when h and σ are sufficiently small, the estimates

$$V - v_2 \leqslant w \leqslant V_1 \tag{2.11}$$

hold for the solution w of (2, 1) to (2, 4). Here V and V_1 are functions constructed in the proof of Lemma 3 of [1], ξ_0 is a constant defined in [1] and

$$v_2 = M_2(\tau + 1)(h + \sigma), \qquad h / \sigma^2 < 1/2\ \nu b_2^2,\ b_2^2 = \max V_1^2$$

Proof. We shall show that Lemma 1 can be applied to the functions $F = V - v_1$ and $F_1 = V_1$ provided that $\tau \leq \tau_0$. It was shown in [1] during the proof of Lemma 2, that

$$\nu (w^{n-1})^2 \frac{\partial^2 V_1}{\partial \eta^2} - \frac{\partial V_1}{\partial \tau} - \eta \frac{\partial V_1}{\partial \xi} + p_x \frac{\partial V_1}{\partial \eta} < 0 \quad \text{in } \Omega$$

$$\nu w^{n-1} \frac{\partial V_1}{\partial \eta} - p_x - v_0 w^{n-1} < 0 \qquad \text{for } \eta = 0$$

and

$$\nu (w^{n-1})^2 \frac{\partial^2 V}{\partial \eta^2} - \frac{\partial V}{\partial \tau} - \eta \frac{\partial V}{\partial \xi} + p_x \frac{\partial V}{\partial \eta} \geqslant 0 \quad \text{in } \Omega$$

$$\nu w^{n-1} \frac{\partial V}{\partial \eta} - p_x - v_0 w^{n-1} > 0 \qquad \text{for } \eta = 0$$

under the assumption that $\tau \leq \tau_0$ and under the condition valid for w^{n-1} only, that $V \leq w^{n-1} \leq V_1$. Hence, provided that h and σ are sufficiently small, M_1 is sufficiently large (M_1 depends on the magnitude of the derivatives of V in the vicinity of $\eta = U(\tau, \xi)$), $\tau = (m+1)h < \tau_0$ and the inequalities (2, 10) hold for $\tau = mh$, the following difference relations

$$L_{m+1}(V - v_1) > 0, \quad \lambda_{m+1}(V - v_1) > 0, \quad L_{m+1}(V_1) < 0, \quad \lambda_{m+1}(V_1) < 0$$

will be fulfilled.

It should be noted that $V_1 > 0$ in Ω' and $V > 0$ everywhere in Ω' except at the surface points $\eta = U(\tau, \xi)$.

Inequalities (2, 10) hold when $\tau = 0$ and, by (2, 2) and according to the properties of V and V_1, when $\xi = 0$. If M_1 is sufficiently large, then the inequality $V - v_1 \leq w$ holds also on $\Gamma_{h\sigma}$ by virtue of the smoothness of V, the latter being equal to zero when $\eta = U(\tau, \xi)$. Since $w = 0$ on $\Gamma_{h\sigma}$, and $V_1 > 0$, hence, obviously $w \leq V_1$ on $\Gamma_{h\sigma}$. Therefore, applying Lemma 1 consecutively to the cases $m = 0, 1, 2, \ldots$ we obtain, that the inequalities (2, 10) hold at all nodes of Ω', for which $\tau \leq \tau_0$.

Inequalities (2, 11) for $\xi \leq \xi_0$ are proved analogously.

Next we shall prove a theorem on convergence of solutions of difference equations (2, 1) to (2, 4), when $h, \sigma \to 0$.

Theorem 1. Let W be solution of the problem (1.4) to (1.6) possessing bounded second derivatives in Ω' and let w be the solution of difference equations (2.1) to (2.4). Then, for sufficiently small h and σ, we have

$$|W - w| \leqslant M_3 (h + \sigma) \tag{2.12}$$

in Ω', provided that either $t_0 \leq \tau_0$ and $h/\sigma^2 < \frac{1}{2}\nu b_1^2$, or $x_0 \leq \xi_0$ and $h/\sigma^2 < \frac{1}{2}\nu b_2^2$.

Proof. Let $X_{mlk} = w_{mlk} - W_{mlk}$. In accordance with the boundary conditions (1.5) and Equations (2.2) and (2.4), we have

$$X_{0lk} = 0, \quad X_{m0k} = 0, \quad X_{mlk} = O(h + \sigma) \quad \text{on } \Gamma_{h\sigma}$$

Boundary condition (1.6) and smoothness of W imply that

$$\nu W_{ml0} \frac{W_{m+1,l1} - W_{m+1,l0}}{\sigma} - p_{xml0} - v_{0ml0} W_{ml0} = O(h + \sigma)$$

Therefore, with (2.3) taken into account, we obtain

$$\nu w_{ml0} \frac{X_{m+1,l1} - X_{m+1,l0}}{\sigma} - v_{0ml0} X_{ml0} + \nu \frac{W_{m+1,l1} - W_{m+1,l0}}{\sigma} X_{ml0} = 0 \tag{2.13}$$

At the internal nodes of Ω' we have, by virtue of the assumption of smoothness of W and by (1.4)

$$[\nu (W_{mlk})^2 + M\sigma] \frac{W_{ml,k+1} - 2W_{mlk} + W_{ml,k-1}}{\sigma^2} - \frac{W_{m+1,lk} - W_{mlk}}{h} -$$
$$- k\sigma \frac{W_{mlk} - W_{m,l-1,k}}{\sigma} + p_{xmlk} \frac{W_{mlk} - W_{ml,k-1}}{\sigma} = O(h + \sigma)$$

which, on subtraction from the corresponding difference equations (2.1), yield

$$[\nu (w_{mlk})^2 + M\sigma] \frac{X_{ml,k+1} - 2X_{mlk} + X_{ml,k-1}}{\sigma^2} - \frac{X_{m+1,lk} - X_{mlk}}{h} -$$
$$- k\sigma \frac{X_{mlk} - X_{m,l-1,k}}{\sigma} + p_{xmlk} \frac{X_{mlk} - X_{ml,k-1}}{\sigma} +$$
$$+ \nu (w_{mlk} + W_{mlk}) \frac{W_{ml,k+1} - 2W_{mlk} + W_{ml,k-1}}{\sigma^2} X_{mlk} = O(h + \sigma) \tag{2.14}$$

Let us now introduce into (2.13) and (2.14) a new function

$$Y_{mlk} = X_{mlk} e^{M_4 k\sigma} \qquad \left(M_4 > \frac{\max |v_0| + \nu \max |\partial W / \partial \eta| + 1}{\nu \min V(\tau, \xi, 0)}\right)$$

where $M_4 = \text{const} > 0$. Obviously

$$Y_{0lk} = 0, \qquad Y_{m0k} = 0, \qquad Y_{mlk} = O(h + \sigma) \text{ on } \Gamma_{h\sigma} \tag{2.15}$$

Equations (2.13) yield

$$\nu w_{ml0} e^{-M_4\sigma} \frac{Y_{m+1,l1} - Y_{m+1,l0}}{\sigma} + A_1 Y_{m+1,l0} + A_2 Y_{ml0} = O(h + \sigma) \tag{2.16}$$

$$A_1 = \nu w_{ml0} \frac{e^{-M_4\sigma} - 1}{\sigma}, \qquad A_2 = \nu \frac{W_{m+1,l1} - W_{m+1,l0}}{\sigma} - v_{0ml0}$$

and by virtue of the choice of M_4, we have

$$|A_1| - |A_2| > 1 \tag{2.17}$$

for sufficiently small σ.

From (2.14) we find, that, for internal nodes of Ω' with coordinates $((m+1)h, l\sigma, k\sigma)$, we have

$$e^{-M_4\sigma}[\nu(w_{mlk})^2+M\sigma]\frac{Y_{ml,k+1}-2Y_{mlk}+Y_{ml,k-1}}{\sigma^2}-\frac{Y_{m+1,lk}-Y_{mlk}}{h}-$$
$$-k\sigma\frac{Y_{mlk}-Y_{m,l-1,k}}{\sigma}+\left(p_{xmlk}-2[\nu(w_{mlk})^2+M\sigma]\frac{1-e^{-M_4\sigma}}{\sigma}\right)\frac{Y_{mlk}-Y_{ml,k-1}}{\sigma}+$$
$$+B_1Y_{ml,k-1}+B_2Y_{mlk}=O(h+\sigma)$$

where

$$B_1=p_{xmlk}\frac{1-e^{M_4\sigma}}{\sigma}+[\nu(w_{mlk})^2+M\sigma]\frac{e^{M_4\sigma}-2+e^{-M_4\sigma}}{\sigma^2}$$
$$B_2=\nu(w_{mlk}+W_{mlk})\frac{W_{ml,k+1}-2W_{mlk}+W_{ml,k-1}}{\sigma^2}$$

These yield the following expression for $Y_{m+1,lk}$.

$$Y_{m+1,lk}=\left(1-\frac{h[(2\nu(w_{mlk})^2+2M\sigma)e^{-M_4\sigma}-p_{xmlk}\sigma+k\sigma^2]}{\sigma^2}-\right.$$
$$\left.-\frac{2[\nu(w_{mlk})^2+M\sigma](1-e^{-M_4\sigma})h}{\sigma^2}\right)Y_{mlk}+\frac{e^{-M_4\sigma}[\nu(w_{mlk})^2+M\sigma]h}{\sigma^2}Y_{ml,k+1}+$$
$$+khY_{m,l-1,k}+\left(\frac{[\nu(w_{mlk})^2+M\sigma]e^{-M_4\sigma}h}{\sigma^2}-\frac{p_{xmlk}h}{\sigma}+\right.$$
$$\left.+\frac{2[\nu(w_{mlk})^2+M\sigma](1-e^{-M_4\sigma})h}{\sigma^2}\right)Y_{m,l,k-1}+B_1hY_{ml,k-1}+B_2hY_{mlk}+hO(h+\sigma) \tag{2.18}$$

Obviously, the sum of all coefficients of Y in the first four terms of the right-hand side of (2.18) is equal to unity. Since

$$h/\sigma^2<1/2\nu b_1^2,\quad h/\sigma^2<1/2\nu b_2^2,\quad M>\max|p_x|$$

it can easily be confirmed that for sufficiently small σ all these coefficients are nonnegative.

Let us now denote by P_m the maximum of $|Y|$ when $\tau\le mh$. Then, either $P_{m+1}=P_m$ or $\max|Y|$ with $\tau\le(m+1)h$ is reached when $\tau=(m+1)h$. If the latter is found to be the internal node of Ω' then from (2.18) it follows that

$$P_{m+1}\leqslant P_m+M_5hP_m+M_6h(h+\sigma)$$

If $\max|Y|$ when $\tau=(m+1)h$ is reached when $\eta=0$ or on $\Gamma_{h\sigma}$, then from (2.15),(2.16) and (2.17) it follows that

$$P_{m+1}\leqslant M_7(h+\sigma)$$

Obviously $P_0=0$. Let us consider the ordinary differential equation

$$ds/d\tau=M_5s+M_6(h+\sigma) \tag{2.19}$$

Clearly, when $mh\le\tau_0$, then P_m does not exceed the solution of (2.17) with the initial condition $s(0)=M_7(h+\sigma)$. Hence, when $\tau\le\tau_0$

$$\max|Y_{mlk}|\leqslant[M_7(h+\sigma)+M_6(h+\sigma)/M_5]e^{M_5\tau_0}-M_6(h+\sigma)/M_5$$

This means that the inequality (2.12) holds and $|W-w|\to0$ as $h,\sigma\to0$, which proves the theorem.

3. Implicit finite difference method. Consider, in the τ, ξ, η-space, a net with nodes defined by the intersection of planes

$$\tau = mh, \quad \xi = lh, \quad \eta = kh, \quad h = \text{const} > 0, \quad m, l, k = 0, 1, 2,$$

We shall call the nodes

$$((m_1 + 1) h, \ l_1 h, \ (k_1 + 1) h), \qquad ((m_1 + 1) h, \ l_1 h, \ (k_1 - 1) h)$$

$$(m_1 h, \ l_1 h, \ k_1 h), \qquad ((m_1 + 1) h, \ (l_1 - 1) h, \ k_1 h)$$

the neighboring nodes of the node $((m_1 + 1) h, \ l_1 h, \ k_1 h)$

As in the previous difference method, we shall call the node belonging to Ω' internal if all its neighboring nodes belong to Ω' (i.e. to the closure of Ω). The remaining nodes of Ω' shall be called boundary nodes. We shall denote the value of f at the point (mh, lh, kh) by f_{mlk} and we shall construct for each internal node of Ω' with coordinates $((m+1)h, lh, kh)$ a corresponding difference equation for w, approximating (1.4)

$$[\nu (w_{mlk})^2 + Mh] \frac{w_{m+1, l, k+1} - 2w_{m+1, lk} + w_{m+1, l, k-1}}{h^2} -$$

$$- \frac{w_{m+1, lk} - w_{mlk}}{h} - kh \frac{w_{m+1, lk} - w_{m+1, l-1, k}}{h} + p_{xmlk} \frac{w_{m+1, lk} - w_{m+1, l, k-1}}{h} = 0$$

$$M = \text{const} > \max |p_x| \tag{3.1}$$

Equations at the boundary nodes

$$w_{0lk} = w_0 (lh, \ kh), \qquad w_{m0k} = w_1 (mh, \ kh), \qquad w_{mlk} = 0 \quad \text{on } \Gamma_h \tag{3.2}$$

correspond to the boundary conditions (1.5) and Γ_h denotes the nodes of Ω' outside the planes $\tau = 0$, $\xi = 0$ and $\eta = 0$. Expression

$$\nu w_{ml0} \frac{w_{m+1, l1} - w_{m+1, l0}}{h} - p_{xml0} - v_{0ml0} w_{ml0} = 0 \tag{3.3}$$

corresponds to the boundary condition (1.6).

Again, before proving the convergence of solutions of the difference system (3.1) to (3.3) to the solution of the problem (1.4) to (1.6) when $h \to 0$, we shall have to establish some auxilliary propositions. First we shall show that the equations of the difference system (3.1) to (3.3) have unique solutions in $w_{m+1,lk}$ under the assumption that all w_{mlk}, are known and that $m \geq 0$ is a fixed integer. This means that the difference equations (3.1) to (3.3) can be solved in successive steps in the τ-direction, i.e. for $m = 0, m = 1, m = 2$, etc.

L e m m a 3. Let $m \geq 0$ be fixed and $w_{ml0} \neq 0$ for all l. Then, the system (3.1) to (3.3) will have a unique solution with respect to $w_{m+1,lk}$, provided that all values of w_{mlk}, i.e. values of w at all nodes for which $\tau = mh$, are known.

P r o o f. Since (3.1) to (3.3) is, for fixed m, a linear algebraic system in $w_{m+1,lk}$ it is sufficient to show that it can have only one solution.

Assume, that for some m, the system (3.1) to (3.3) has two solutions in $w_{m+1, lk}$ and let us denote their difference by $S_{m+1, lk}$. This difference satisfies Equations

$$S_{m+1, 0k} = 0, \qquad S_{m+1, lk} = 0 \quad \text{on } \Gamma_h, \qquad S_{m+1, l1} - S_{m+1, l0} = 0 \tag{3.4}$$

At each internal node of Ω' for which $\tau = (m+1)h$, we have

$$[\nu (w_{mlk})^2 + Mh] \frac{S_{m+1, l, k+1} - 2S_{m+1, lk} + S_{m+1, l, k-1}}{h^2} -$$

$$- \frac{S_{m+1, lk}}{h} - kh \frac{S_{m+1, lk} - S_{m+1, l-1, k}}{h} + p_{xmlk} \frac{S_{m+1, lk} - S_{m+1, l, k-1}}{h} = 0 \tag{3.5}$$

If $S_{m+1,\,lk} \not\equiv 0$, then there exists a point, at which the modulus of $S_{m+1,\,lk}$ assumes its greatest value. By (3.4), $|S_{m+1,l,k}|$ should be reached at some internal node of Ω' with coordinates $((m+1)h,\ l_1 h,\ k_1 h)$. Multiplying (3.5) by $S_{m+1,l,k}$ we obtain

$$\Big\{-[\nu(w_{ml_1k_1})^2+Mh]\frac{S_{m+1,\,l_1,k_1}-S_{m+1,\,l_1,k_1+1}}{h^2}-[\nu(w_{ml_1k_1})^2+Mh-p_{xml_1k_1}h]\times$$
$$\times\frac{S_{m+1,\,l_1k_1}-S_{m+1,\,l_1,\,k_1-1}}{h^2}-k_1h\frac{S_{m+1,\,l_1k_1}-S_{m+1,\,l_1-1,\,k_1}}{h}\Big\}S_{m+1,\,l_1k_1}-$$
$$-\frac{(S_{m+1,\,l_1k_1})^2}{h}=0$$

Obviously, all the terms contained in the left-hand part of this equation should be nonpositive, while the term $(S_{m+1,l_1k_1})^2/h$ will be negative. This is impossible unless $S_{m+1,lk}\equiv 0$, which completes the proof.

Let us now introduce the notation

$$\Lambda_{m+1}(z)\equiv[\nu(w_{mlk})^2+Mh]\frac{z_{m+1,\,l,\,k+1}-2z_{m+1,\,lk}+z_{m+1,\,l,\,k-1}}{h^2}-$$
$$-\frac{z_{m+1,\,lk}-z_{mlk}}{h}-kh\frac{z_{m+1,\,lk}-z_{m+1,\,l-1,\,k}}{h}+p_{xmlk}\frac{z_{m+1,\,lk}-z_{m+1,\,l,\,k-1}}{h}.$$

Lemma 4. Let w given at the nodes of Ω', satisfy the difference equations (3.1) to (3.3) and let the functions Φ and Φ_1 be such, that

$$\Phi\leqslant w\leqslant\Phi_1 \tag{3.6}$$

at the nodes of Ω' for which $\tau = mh$, and also when $l=0$ and on Γ_h, when $\tau=(m+1)h$. Here $m\geq 0$ is a fixed number.

Assume now that at all internal nodes of Ω' for which $\tau=(m+1)h$

$$\Lambda_{m+1}(\Phi)\geqslant 0,\qquad \Lambda_{m+1}(\Phi_1)\leqslant 0$$

at all boundary nodes of the plane $\eta=0$

$$\lambda_{m+1}(\Phi)>0,\qquad \lambda_{m+1}(\Phi_1)<0$$

and let $w_{ml0}\neq 0$ no matter what the value of l is. Then the inequalities (3.6) hold also for $\tau=(m+1)h$.

Proof. We shall first show that $Z=w-\Phi\geq 0$ when $\tau=(m+1)h$. By the condition of the Lemma $Z\geq 0$ when $\tau=mh$, and also when $\xi=0$ and on Γ_h. By (3.3) and the condition $\lambda_{m+1}(\Phi)>0$, we have

$$\nu w_{ml0}\frac{Z_{m+1,\,l1}-Z_{m+1,\,l0}}{h}<0$$

from which it follows that

$$Z_{m+1,\,l0}>Z_{m+1,\,l1} \tag{3.7}$$

For the internal nodes of Ω' lying on the plane $\tau=(m+1)h$, we have

$$\Lambda_{m+1}(w)-\Lambda_{m+1}(\Phi)\leqslant 0$$

which means, that for these nodes

$$-[\nu(w_{mlk})^2+Mh]\frac{Z_{m+1,lk}-Z_{m+1,l,k+1}}{h^2}-\frac{Z_{m+1,lk}-Z_{mlk}}{h}- \tag{3.8}$$
$$-[\nu(w_{mlk})^2+Mh-p_{xmlk}h]\frac{Z_{m+1,lk}-Z_{ml,k-1}}{h^2}-kh\frac{Z_{m+1,lk}-Z_{m+1,l-1,k}}{h}\leqslant 0$$

If $Z_{m+1,lk}$ assumes negative values, then the negative minimum of $Z_{m+1,lk}$ should be achieved at some internal node of Ω', since $Z\geq 0$ when $\xi=0$ and on Γ_h, and the inequality (3.7) holds. All the terms of the left-hand side of (3.8) considered for the point at which the smallest negative value is assumed are nonnegative, and at least one of them is positive, which is impossible. Consequently, $Z_{m+1,lk}\geqslant 0$ everywhere at the nodes of Ω', which was to be proved. In the analogous manner we show that $w-\Phi_1\leq 0$ in Ω' when $\tau=(m+1)h$.

Lemma 5. Lemma 2 holds for the solution w of difference equations (3.1) to (3.3), i.e. with h sufficiently small, inequalities (2.10) when $\tau\leq\tau_0$ and inequalities (2.11) when $\xi\leq\xi_0$, hold.

Proof of this Lemma is analogous to that of Lemma 2.

Now we shall prove the convergence of the difference system (3.1) to (3.3). Here K_i will denote positive constants defined by the parameters of the problem (1.4) to (1.6) and independent of h.

Theorem 2. Let W be the solution of (1.4) to (1.6) possessing bounded second derivatives in Ω' and let w be the solution of difference equations (3.1) to (3.3). Then, at the nodes of Ω' we have, for sufficiently small h,

$$|W-w|\leqslant K_1 h \tag{3.9}$$

provided that either $t_0\leq\tau_0$ or $x_0\geq\xi_0$ where τ_0 and ξ_0 are positive constants defined in Lemma 2.

Proof. Let us denote $w_{mlk}-W_{mlk}$ by X_{mlk}. According to (1.5) and (3.2) we have

$$X_{0lk}=0,\ X_{m0k}=0,\ X_{mlk}=O(h)\quad \text{on } \Gamma_h \tag{3.10}$$

Boundary condition (1.6) and assumption of the smoothness of W imply

$$\nu W_{ml0}\frac{W_{m+1,l1}-W_{m+1,l0}}{h}-p_{xml0}-v_{0ml0}W_{ml0}=O(h)$$

Hence, taking (3.3) into account, we obtain

$$\nu w_{ml0}\frac{X_{m+1,l1}-X_{m+1,l0}}{h}+\left(-v_{0ml0}+\nu\frac{W_{m+1,l1}-W_{m+1,l0}}{h}\right)X_{ml0}=O(h) \tag{3.11}$$

for all boundary nodes lying in the plane $\eta=0$. For internal nodes of Ω' we have, by (1.4) and the smoothness of W

$$[\nu(W_{mlk})^2+Mh]\frac{W_{m+1,l,k+1}-2W_{m+1,lk}+W_{m+1,l,k-1}}{h^2}-$$
$$-\frac{W_{m+1,lk}-W_{mlk}}{h}-kh\frac{W_{m+1,lk}-W_{m+1,l-1,k}}{h}+$$
$$+p_{xmlk}\frac{W_{m+1,lk}-W_{m+1,l,k-1}}{h}=O(h)$$

which, together with (3.1), yield

$$[\nu(w_{mlk})^2 + Mh] \frac{X_{m+1,\,l,\,k+1} - 2X_{m+1,\,lk} + X_{m+1,\,l,\,k-1}}{h^2} - \frac{X_{m+1,\,lk} - X_{mlk}}{h} -$$
$$- kh \frac{X_{m+1,\,lk} - X_{m+1,\,l-1,\,k}}{h} + p_{xmlk} \frac{X_{m+1,\,lk} - X_{m+1,\,l,\,k-1}}{h} +$$
$$+ \frac{W_{m+1,\,l,\,k+1} - 2W_{m+1,\,lk} + W_{m+1,\,l,\,k-1}}{h^2} \nu(w_{mlk} + W_{mlk}) X_{mlk} = O(h) \quad (3.12)$$

Let us now introduce into (3.10), (3.11) and (3.12), a new function

$$Y_{mlk} = X_{mlk} e^{M_4 kh - K_2 mh} \qquad (M_4 > 0,\ K_2 > 0)$$

where M_4 is defined in the proof of Theorem 1 and the constant K_2 is defined below. Equations (3.11) yield

$$\nu w_{ml0} \frac{Y_{m+1,\,l1} - Y_{m+1,\,l0}}{h} e^{-M_4 h} + C_1 Y_{m+1,\,l0} + C_2 Y_{ml0} = O(h) \qquad (3.13)$$

$$C_1 = \nu w_{ml0} \frac{e^{-M_4 h} - 1}{h}, \qquad C_2 = e^{-K_2 h}\left(-v_{0ml0} + \nu \frac{W_{m+1,\,l1} - W_{m+1,\,l0}}{h}\right)$$

For internal nodes of Ω', (3.12) give

$$[\nu(w_{mlk})^2 + Mh] e^{-M_4 h} \frac{Y_{m+1,\,l,\,k+1} - 2Y_{m+1,\,lk} + Y_{m+1,\,l,\,k-1}}{h^2} -$$
$$- e^{-K_2 h} \frac{Y_{m+1,\,lk} - Y_{mlk}}{h} - kh \frac{Y_{m+1,\,lk} - Y_{m+1,\,l-1,\,k}}{h} +$$
$$+ \left(p_{xmlk} - 2[\nu(w_{mlk})^2 + Mh] \frac{(1 - e^{-M_4 h})}{h}\right) \frac{Y_{m+1,\,lk} - Y_{m+1,\,l,\,k-1}}{h} +$$
$$+ D_1 Y_{m+1,\,l,\,k} + D_2 Y_{mlk} + D_3 Y_{m+1,\,l,\,k-1} = O(h) \qquad (3.14)$$

$$D_1 = -\frac{1 - e^{-K_2 h}}{h}$$

$$D_2 = \nu(w_{mlk} + W_{mlk}) \frac{W_{m+1,\,l,\,k+1} - 2W_{m+1,\,lk} + W_{m+1,\,l,\,k-1}}{h^2} e^{-K_2 h}$$

$$D_3 = p_{xmlk} \frac{1 - e^{M_4 h}}{h} + [\nu(w_{mlk})^2 + Mh] \frac{e^{M_4 h} - 2 + e^{-M_4 h}}{h^2}$$

which can be written as

$$-[\nu(w_{mlk})^2 + Mh] e^{-M_4 h} \frac{Y_{m+1,\,lk} - Y_{m+1,\,l,\,k+1}}{h^2} - [[\nu(w_{mlk})^2 + Mh] e^{-M_4 h} -$$
$$- p_{xmlk} h + 2[\nu(w_{mlk})^2 + Mh](1 - e^{-M_4 h})] \frac{Y_{m+1,\,lk} - Y_{m+1,\,l,\,k-1}}{h^2} -$$
$$- e^{-K_2 h} \frac{Y_{m+1,\,lk} - Y_{mlk}}{h} - kh \frac{Y_{m+1,\,lk} - Y_{m+1,\,l-1,\,k}}{h} + D_1 Y_{m+1,\,lk} +$$
$$+ D_2 Y_{mlk} + D_3 Y_{m+1,\,l,\,k-1} = O(h) \qquad (3.15)$$

Assume that the modulus of Y assumes its greatest value at the point P whose coordinates are $((m+1)h,\ l_1 h,\ k_1 h)$. If P lies on Γ_h, on the plane $\tau = 0$ or on $\xi = 0$, then from (3.10) it follows that $|Y_{m_1+1,l_1,k_1}| \leqslant K_3 h$. If P lies on $\eta = 0$ then, by virtue of the choice of M_4 it follows from (3.13) that $|Y_{m_1+1,l_1,k_1}| \leqslant K_4 h$ for sufficiently small h, since $|C_1| - |C_2| > 1$ if h is sufficiently small.

If, on the other hand, P is an internal node of Ω', then first four terms of the right-hand side of (3.15) considered at the point P are of the same sign coinciding with the sign of the fifth term, provided that

$$(\nu (w_{mlk})^2 + Mh)\, e^{-M_4 h} - p_{xmlk} h + 2\, [\nu (w_{mlk})^2 + Mh]\,(1 - e^{-M_4 h}) \geqslant 0 \qquad (3.16)$$

Inequality (3.16) will be fulfilled for sufficiently small h, since by the previous assumption $M > \max |p_x|$. Hence, at the point P

$$(|D_1| - |D_2| - |D_3|)\,|Y_{m_1+1,l_1k_1}| \leqslant K_5 h \qquad (3.17)$$

Let us choose K_2 large enough to fulfil the inequality

$$K_2 > \nu(\max V_1 + \max W) \max \left|\frac{\partial^2 W}{\partial \eta^2}\right| + \max |p_x|\, M_4 + \max V_1^2 M_4^2 \nu$$

Then, the coefficient of Y_{m_1+1,l_1,k_1} in the left-hand side of the inequality (3.17) is positive and $|Y_{m_1+1l_1,k_1}| \leqslant K_6 h$. Consequently, for sufficiently small h the inequality (3.9) holds for $w - W$, which completes the proof.

4. Construction of approximate solution of the problem (1.1) to (1.3). We can find approximate values of the function $u(t, x, y)$ defined by the system (1.1) to (1.3), using the approximate representation of the inverse of $u(t, x, y)$ in terms of w_{mlk} with fixed t and x

$$y = \sum_{k=0}^{[u/h]} \frac{h}{w(t, x, kh)} \qquad (4.1)$$

which, together with (1.7), readily yields the result that in the region D when $y \leq y_0 < \infty$ and when either $t_0 \leq T_0$ or $x_0 \leqslant \xi_0$, $|u_{mlk} - u| \leqslant K_7 h$.

Here K_7 depends on y_0, function u_{mlk} is defined by (4.1) and w is the solution of difference equations (3.1) to (3.3). Estimates obtained for w_{mlk} in Lemma 5 should be taken here into account together with the estimates for W obtained in [1] in Lemmas 2 and 3. Analogous statement is correct for solutions of the difference system (2.1) to (2.4).

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